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Transition to chaos through a converging sequence of maps

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Abstract. We consider an infinite sequence or one-parameter family of 1D maps $\{f_m(x)\}$ which converges uniformly to a target map $f(x)$. The parameter m can be continuous or discrete. $f_m(x)$ have non-linear dependence on m. The target map $f(x)$ generates fully chaotic dynamics. We analyse the dynamics generated by the sequence $\{f_m\}$ as the target map $f(x)$ is approached through this sequence. Here we present the first results of computer experiments regarding two special cases of the above situation.

The intuitive background of this paper is as follows. Most of the non-linear maps used to describe chaotic dynamics depend on one or more control parameters. When the control parameters are varied continually over a range of values, presumably by a controlling physical agency external to the system, the system goes through various stages of regular and/or chaotic dynamics [1-6]. In this model the maps governing the dynamics generally depend linearly on the control parameter. This approach is very well suited for describing laboratory experiments where the control parameters are essentially the experimental parameters. However, for the natural occurrence of chaotic motion the role of control parameters may become quite superfluous. It is more likely that the parameters which are responsible for the dynamical evolution of a system depend in a complicated way on the dynamical state of the system, as well as on the external environment. To see this point we compare the turbulent flow in some part of the Earth's atmosphere with the Rayleigh-Bénard experiment [7, 8] in a laboratory. In the former case, there can neither be constant temperature gradients across the system, nor can these gradients be varied in any controlled fashion. The temperature gradients, for the natural turbulent flow, vary with space and time. The distribution of the temperature gradients and the velocity field of the fluid depend on each other in a complicated way. As one more example, the dynamics of the human brain is obviously influenced by external impulses, but their relation is known to be extremely complicated and non-linear [9]. With this point of view we thought it worthwhile to explore the possibilities of dynamical evolution of a system depending on naturally changing external conditions which make the system chaotic (e.g. a river approaching a fall).

Henceforth we consider only 1D dynamical systems with discrete time.

There are two types of 1D maps which give rise to different types of transitions from order to chaos. First is the class of unimodal maps which are differentiable maps having only one extremum over their domain of definition and which are symmetric about this extremum [10]. The most famous example of this class of maps is the logistic map $L_\mu(x) = \mu x(1-x)$ $(0 < \mu \leq 4)$. As the parameter μ is varied continuously from 0 onwards, the chaotic behaviour sets in through infinite cascades of period doubling bifurcations for $\mu > \mu_\infty = 3.5700\ldots$ [1, 3]. The range $\mu_\infty < \mu \leqslant 4$ is densely covered by the intervals (or windows) in μ values which give rise to regular dynamics (attractive periodic orbits). Within each such window, an infinite bifurcation cascade occurs, whose smallest cycle has an odd period. If μ is decreased through the point at which such a window begins, the system makes a transition to chaos, which is called the intermittent transition [11]. Here, in the transition region, the chaotic character appears intermittently in time. The set of values of μ for which chaos occurs does not form any intervals although it has a positive Lebesgue measure [4, 12–14].

The other class of 1D maps generating chaotic dynamics is represented by the supercritical Newton iteration near a double root [15]. This is driven by a single parameter α/γ such that $\alpha/\gamma<0$, $=0$ and >0 correspond to subcritical (regular), critical and supercritical (chaotic) dynamics, respectively. Written explicitly, we get

$$x_{t+1} = \begin{cases} (x_t^2+1)/2x_t & \text{subcritical} \\ x_t/2 & \text{critical} \\ (x_t^2-1)/2x_t & \text{supercritical.} \end{cases}$$

Thus the transition to chaos occurs at a single value of the parameter. The trajectories of the supercritical Newton iteration can be obtained from those of the so-called Bernoulli shift over two symbols:

$$x_{t+1} = 2x_t \pmod 1 \qquad 0 \leqslant x \leqslant 1$$

by a simple transformation. The Lyapounov number (see below) of this map is a constant independent of x and of any parameter. Its value is ln 2. Similarly, transition to chaos occurs in the so-called tent map:

$$F(x) = a(1-2|x-1/2|) \qquad 0 < a \leqslant 1$$

at $a=\frac{1}{2}$ [3]. Here the Lyapounov number changes logarithmically with the parameter a and is given by $\ln(2a)$ which matches with that of the Bernoulli shift for $a=1$.

With the above intuitive background and a very brief summary of the transitions to chaos that have been observed for various 1D maps, we analyse two special cases of the following situation. Consider an infinite sequence or one-parameter family of non-linear maps $\{f_m(x)\}$, pertaining to a 1D dynamical system where $f_m: I \to I$ maps an interval I into itself. Each $f_m(x)$ generates a possible dynamics of the system. Further, each f_m depends non-linearly on the parameter m which labels the sequence. $\{f_m(x)\}$ converges uniformly to a map $f(x)$ which is non-linear and does not depend on m. The convergence must be at least in the C^0 sense, i.e. in the sup norm,

$$d(f_m, f) = \sup\{|f_m(x) - f(x)|;\ x \in I\}.$$

We then analyse how the dynamics of the system changes as the sequence $\{f_m(x)\}$ is scanned to approach the target map $f(x)$. In the two special cases considered we have chosen the target map $f(x)$ to generate fully chaotic dynamics of the system.

In order to know whether a system is chaotic or regular we make use of the Lyapounov numbers [4, 16]. For 1D maps the Lyapounov number is defined as

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log_2 |f'(f^k(x_0))| \tag{1}$$

where $f^k(x_0)$ is the kth iterate of f evaluated at x_0 and the prime denotes differentiation WRT x. For a 1D system, if an invariant ergodic probability measure ρ WRT f exists, then the above limit exists and is the same for ρ almost all x_0 [4, 16]. Further, if more than one attractors coexist, with open intervals as their basins of attraction, then a Lyapounov number can be separately defined for each of these basins. It is well known that the positivity of Lyapounov number corresponds to the sensitivity to the initial conditions [1, 4], i.e. to the chaotic nature of the dynamics. The actual formula used to compute the Lyapounov number is [17]

$$\lambda = \frac{1}{100\,000} \sum_{k=1}^{100\,000} \ln|f'(f^{k-1}(x_0))|/\ln 2. \tag{2}$$

While computing the above sum, the first 1000 iterates were ignored so that the orbit settles down to the corresponding attractor.

Case I. Our first case comprises a one-parameter family of maps given by

$$f_m(x) = \{\tfrac{3}{2}[1-\tfrac{1}{2}m\cos(4\pi x)\sin(2/m)]+1\}x(1-x) \qquad 0 \leqslant x \leqslant 1. \tag{3}$$

For any real m $(-\infty < m < \infty)$, except for $m = 0$, $f_m(x)$ maps the interval [0, 1] into itself. However, for any real m, $f_m(x)$ is invariant under the change of sign of m so that $f_m(x)$ and $f_{-m}(x)$ generate identical dynamics. Further, for $m = 0$, the map is not defined. Thus one has to consider only the range $0 < m < \infty$. It is easy to see that, as $m \to \infty$, the one-parameter family of maps given by (3) converges uniformly to the map

$$f(x) = [3\sin^2(2\pi x)+1]x(1-x). \tag{4}$$

This convergence is obvious from figures 1(*a*) and (*b*) where the convergence is apparent even at $m = 5$. In fact, this convergence is C^∞.

We note that the maps $f_m(x)$ are not unimodal and a large number of general results obtained for the unimodal maps [10] are not available for $f_m(x)$. Maps $f_m(x)$ and the target map $f(x)$ are bimodal and are symmetric about the unique minimum at $x = \frac{1}{2}$. Figure 1 shows that the target map $f(x)$ has three fixed points, two of them being unstable ($|f'(x^*)| > 1$ where x^* is the fixed point). At $x = 0$, $|f'(0)| = 1$ so that it is critically stable. The above statements also hold for all $f_m(x)$ with m exceeding about 1.3.

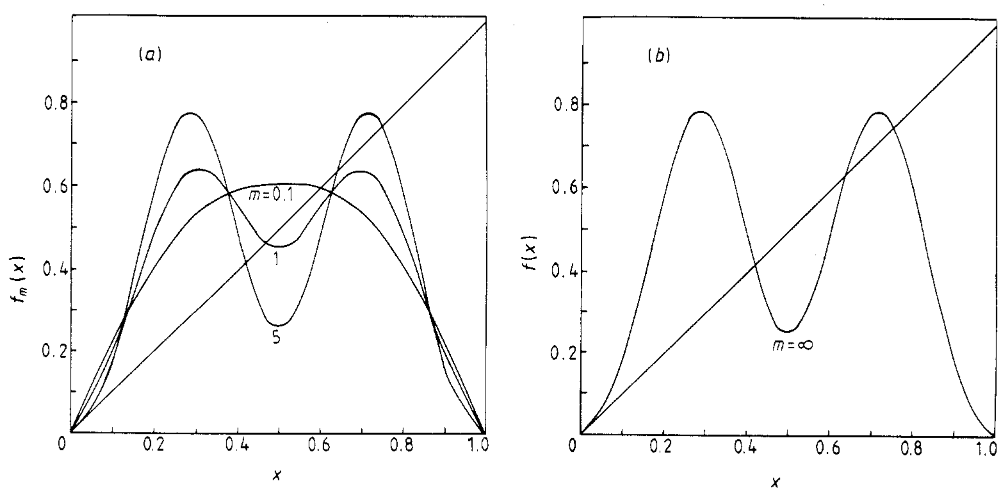

Figure 1. (*a*) Maps $f_m(x)$ (equation (3)) are plotted for initial values of m. (*b*) The target map $f(x)$ (4). Convergence of the sequence $\{f_m(x)\}$ to the target map $f(x)$ is clearly seen.

Assuming the time to be discrete we can now simulate the dynamics by computing the time series for any given f_m as

$$x_{n+1} = f_m(x_n) \qquad n = 0, 1, 2, \ldots$$

or

$$x_n = f_m^n(x_0) \qquad 0 < x_0 < 1.$$

We now give the outline of the computer experiments performed. The computations were done using quadruple precision arithmetic giving a maximum precision of 36 decimal digits. The intervals in m corresponding to the the attractive periodic orbits were identified by varying m at a step of 0.001. The initial point of the orbit, x_0, was systematically varied to check whether two or more attractors coexist for the same value of m. To find out the location (m value) at which successive cycles appear, power spectra of the time series were extensively used. Parallel to this processing Lyapounov numbers were computed for $0 < m \leq 101$ with a step of 0.001. Positivity of Lyapounov numbers was taken as a signature of chaos.

The principal findings of the computer experiments are as follows.

The dynamics generated by (4) is fully chaotic consistent with a Lyapounov number $\lambda_\infty = 0.832\,789$, provided the initial point of the trajectory is greater than about 0.01. For $x_0 < 0.01$, λ_∞ takes a very small negative value (~ -0.0001) and a very long period cycle seems to set in.

There are four major findings regarding the dynamics generated by the sequence of maps $\{f_m(x)\}$.

(a) As m increases from zero onwards we first get a sequence of cycles with periods 2^n with basic period $k = 1$. However, no cycle with period >128 was detected in this cascade, till the first transition to chaos, which occus at about 1.224 680, was reached. The minimum step size in m used here was 10^{-6}.

(b) As m increases further, we get a finite number of finite windows in m corresponding to attractive periodic orbits. Each window corresponds to one or more basic periods (see (c)) and bifurcation cascades. These windows are separated by intervals in m values corresponding to the chaotic states of the system (however, see (c)). For $m > 39$, no bifurcation cascades are detected. There are two non-degenerate stable cycles with basic period unity giving rise to the 2^n bifurcation cascade. None of the bifurcation bascades was found to contain a cycle with period >256 within a resolution of 10^{-5} in m. An interesting and important finding regarding all the observed bifurcation cascades is that almost every bifurcation is interrupted by a long period cycle (period >8192). Thus the observed bifurcations are not sharp. The reason for this lies in the fact that the derivative of $f_m(x)$ does not change monotonically with m. This also seems to be the reason why we do not observe infinite bifurcation cascades within $\Delta m = 10^{-6}$.

(c) For $m > 1.3$ or so, there are intervals of m values for which both the regular and chaotic dynamics occur simultaneously. In this case the total domain [0, 1] gets divided into two groups of subintervals. One group forms the basin of attraction for the periodic attractor (actually a fixed point) and the other group forms the basin of attraction for the chaotic attractor. This phenomenon also occurs for windows where the two different attractors are periodic with different basic periods, one of which is again a fixed point.

(d) Figures 2(a) and (b) show the dependence of Lyapounov numbers, λ_m, on m. Whenever the regular and chaotic dynamics coexisted, the initial point x_0 in (2) belonged to the basin of attraction of the chaotic attractor. One can easily see that, while λ_m is a discontinuous and fluctuating function of m for small m, as m increases,

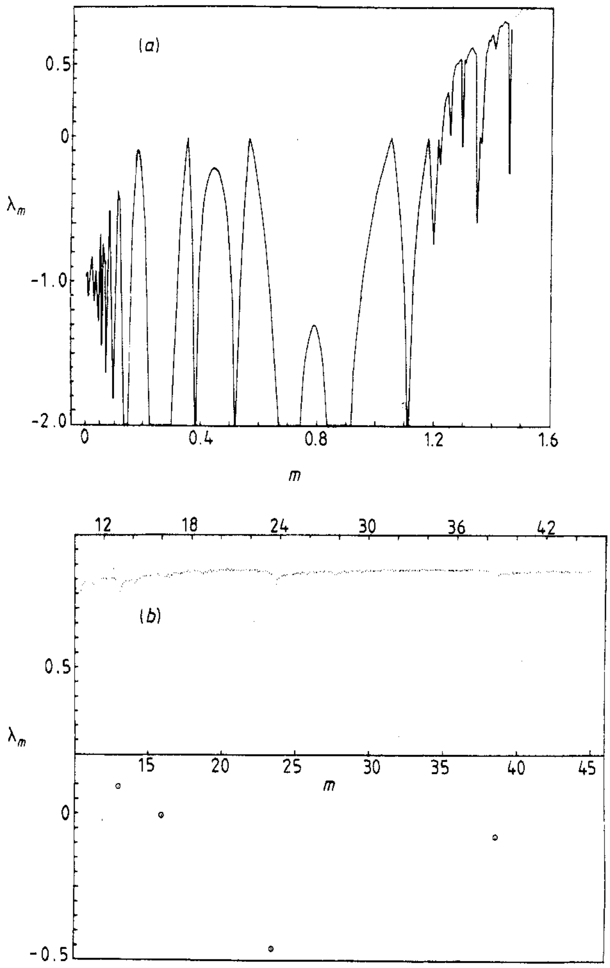

Figure 2. Lyapounov numbers λ_m corresponding to $f_m(x)$ in (3) are plotted (a) for some initial m values and (b) for $10 \leqslant m \leqslant 45$. The clustering tendency of λ_m around an average value is quite apparent. (For details see text.)

λ_m cluster around some average value, with very small fluctuations. Lyapounov numbers were computed for $0.001 \leqslant m \leqslant 101.0$ with a step size of 0.001. It was found that, for $39.0 < m \leqslant 101.0$, λ_m never become negative. Their average value over the range $60.0 \leqslant m \leqslant 101.0$ is 0.834 650, rounded to six digits. This average value is very close to the Lyapounov number λ_∞ corresponding to the target map of (4). Further, the convergence of $\{f_m\}$ (3) to $f(x)$ (4) is pretty fast (see figure 1) and the limit $m \to \infty$ can be assumed to have been realised in practice for $m = 101.0$. In order to check on this assertion, we compressed the range of m $(0, \infty)$ to $(0, 4]$ using the transformed parameter

$$m = \tan(\pi a/8).$$

For $m = 101.0$ $a \simeq 3.9748$, which is fairly close to 4. To check on the further behaviour of λ_m we computed λ_m for values of m ranging up to $m = 10\,001.0$. These are summarised in table 1. We see that for $m > 1001.0$, λ_m stay very close to λ_∞. However, we have no proof of the convergence of the sequence $\{\lambda_m\}$ to λ_∞.

On the basis of the above observations, we can say that, as $m \to \infty$, the sequence $\{\lambda_m\}$ stays within a small neighbourhood of λ_∞. Further, if r is the size of the smallest of such neighbourhoods, then $r \ll |\lambda_m|$ and $r \ll |\lambda_\infty|$ which means that the sequence $\{\lambda_m\}$ clusters around λ_∞. Since λ_m $(m > 39)$ and λ_∞ are both positive, we conclude that, as $\{f_m\}$ is scanned towards the target map $f(x)$, after a finite m value, the maps $f_m(x)$ and the target map $f(x)$ share the property of generating chaotic dynamics. Further, after a finite m value, the chaotic character of the system dynamics becomes stable, in fact, becomes almost independent of m.

For the case of the logistic map $L_\mu(x) = \mu x(1-x)$ $(1 \leq \mu \leq 4)$, the qualitative features of the period doubling bifurcations can be obtained using local analysis in the following way [15]. To start with one must know the expression for the (non-trivial) fixed point of $L_\mu(x)$ in terms of μ and the value of μ at which the fixed point of $L_\mu(x)$ becomes unstable. These are $x^* = 1 - \mu^{-1}$ and $\mu = 3$ which are obtained, respectively, by solving the equations

$$L_\mu(x) = x$$

and

$$|L'_\mu(x^*)| = 1.$$

In order to construct the 2-cycle occurring just after $\mu = 3$, we expand the function $g_\mu(x) = L^2_\mu(x^* - x) - x^*$ $(g_\mu(0) = 0)$, about $x = 0$ in Taylor's series up to third order to get a cubic polynomial in x whose coefficients are polynomials in μ. The expression for $g_\mu(x)$ on the higher side of the critical parameter value $\mu = 3$ is obtained by putting $\mu = 3 + \varepsilon$ $(\varepsilon > 0)$ and keeping the first-order terms in ε. Finally, the 2-cycle $\{x_1^*, x_2^*\}$ is obtained by solving $g_{3+\varepsilon}(x) = x$. The next bifurcation is obtained by iteratively repeating the above procedure.

For the maps $f_m(x)$ in equation (3)

$$f_m(x) = x$$

is a transcendental equation and defies any closed form solution. Hence one cannot take the above analytical approach without resorting to numerical methods. Thus numerical methods seem to be essential for the analysis of the dynamics generated by these maps.

Table 1. Averages of the Lyapounov numbers computed over the unit intervals of m values exceeding $m = 1000$. The value of λ_∞ is included for comparison. The step size in m is 0.001.

Interval in m	Average (Rounded to six digits)
1 001.0–1 002.0	0.833 062
2 001.0–2 002.0	0.832 745
3 001.0–3 002.0	0.832 861
4 001.0–4 002.0	0.832 809
5 001.0–5 002.0	0.832 787
6 001.0–6 002.0	0.832 781
7 001.0–7 002.0	0.832 552
10 001.0–10 002.0	0.832 552
λ_∞	0.832 789

Further, the dynamics of the logistic map after the first transition to chaos is analysed in terms of the so-called U sequences and the analysis crucially depends on the fact that these maps are unimodal [5, 6, 10]. Since $f_m(x)$ are bimodal, they do not yield to such analysis. Thus the dynamics generated by $f_m(x)$ seems to lack a coherent mathematical model. Thus, for example, a satisfactory explanation of the observed dependence of λ_m on m ((figure 2) in terms of the properties of f_m seems to be a mathematically difficult research problem.

In spite of the above considerations, our experimental findings clearly indicate that the dynamical evolution through the converging sequence of maps $\{f_m(x)\}$ is qualitatively different from the well known dynamical evolution of the unimodal 1D maps. To see this we make the following observations.

(i) Our experiments have shown the existence of *intervals* in the parameter (m) values for which the Lyapounov number >0. For the case of logistic type of maps the set of values of the control parameter, μ, does not form any intervals although this set has a positive Lebesgue measure [4, 12–14]. Thus these maps are not even topologically equivalent [10].

(ii) We have seen only a finite number of windows corresponding to the attractive periodic orbits, while the corresponding windows for the logistic type of maps are infinite and are dense over the whole range $\mu_\infty < \mu \leq 4$ [1, 3–6, 12–14]. Note that these two observations are consistent with each other.

(iii) After a finite value of m the Lyapounov number is always positive, becomes independent of the parameter m and stabilises in a small neighbourhood around λ_∞. This observation does not seem to have any analogue in the case of unimodal maps. In fact, for the logistic map the iterates in the chaotic regime jump between 2^n subintervals (or islands) of the interval $[0, 1]$ with n decreasing from ∞ to 0 as μ varies from μ_∞ to 4. This so-called reverse or period halving bifurcation sequence of chaotic bands is not found for $\{f_m(x)\}$.

In order to see whether the Lyapounov number is independent of the initial point of the trajectory or not, we computed Lyapounov numbers for $f(x)$ in (4) for 100 different values of x_0 in (2). The 100 values for x_0 were generated using a uniform random number generator, giving random numbers between 0.01 and 1. The Lyapounov numbers turned out to be constant with average $\lambda_\infty = 0.832\,789$ and standard deviation $0.152\,632 \times 10^{-2}$ both rounded to six digits.

Case II. Here we consider a discrete sequence of maps:

$$\phi_m(x) = \left(1 + \sum_{k=1}^{m} (-1)^k x^k\right)(1 - 2|x - 1/2|) \qquad 0 \leq x \leq 1;\ m = 1, 2, \ldots. \tag{5}$$

Each $\phi_m(x)$ maps the interval $[0, 1]$ into itself. Again, it is easy to see that, as $m \to \infty$, the sequence $\{\phi_m(x)\}$ converges uniformly to the map

$$\phi(x) = \left(\frac{1}{1+x}\right)(1 - 2|x - \tfrac{1}{2}|). \tag{6}$$

This fact can be seen from figures 3(a) and (b) which depict the first few $\phi_m(x)$ and the target map $\phi(x)$. Note that ϕ_m are asymmetric functions and are not differentiable at $x = \frac{1}{2}$. This does not cause any difficulty in computing the Lyapounov number for a chaotic attractor ($\lambda_m > 0$) because the probability that any iterate on a chaotic attractor exactly equals $\frac{1}{2}$ is zero.

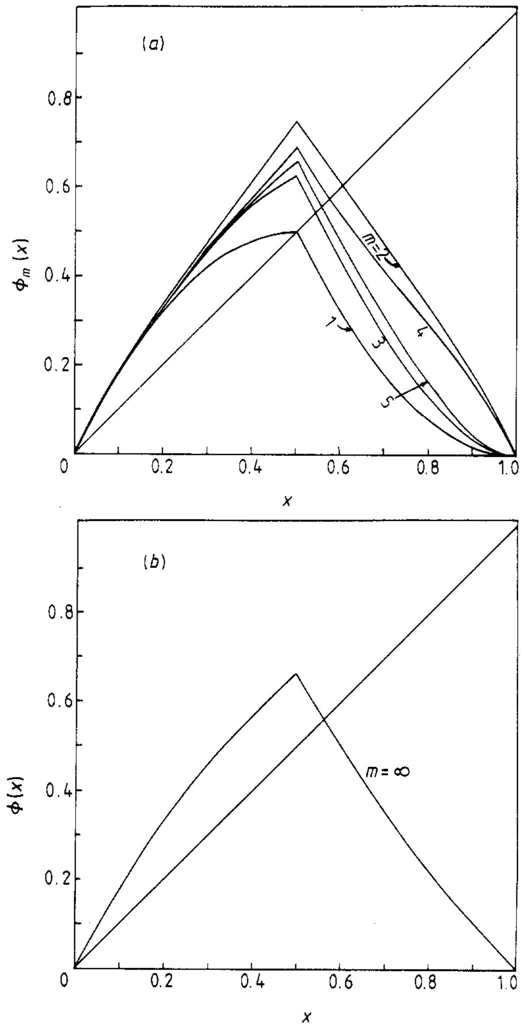

Figure 3. (*a*) Maps $\phi_m(x)$ (5) are plotted for some initial m values. (*b*) The target map $\phi(x)$ (6). Again, the convergence of $\{\phi_m(x)\}$ to $\phi(x)$ is obvious. Note the oscillatory behaviour of $\phi_m(x)$.

Figures 3(*a*) and (*b*) show that $\phi(x)$, $\phi_m(x)$ have two fixed points, both of them unstable. Note that $x=0$ is an unstable fixed point. However, $x=1$ is mapped onto $x=0$ for all m.

The dynamics due to the target map $\phi(x)$ (6) is fully chaotic with Lyapounov number $\lambda_\infty = 0.382\,91$, rounded to five digits. It has a single chaotic attractor with open interval $(0, 1)$ as its basin of attraction.

The dynamics generated by the sequence of maps $\phi_m(x)$ is also very simple. For $m=1$ the whole of the open interval $(0, 1)$ is attracted to the fixed point $x=\frac{1}{2}$. So the attractor consists of a single fixed point. For $m>1$ the dynamics is chaotic with Lyapounov numbers λ_m all exceeding zero.

Figure 4 displays the dependence of λ_m on the parameter m. One can see that, as m increases from two onwards, λ_m oscillates around some mean value. Initially the

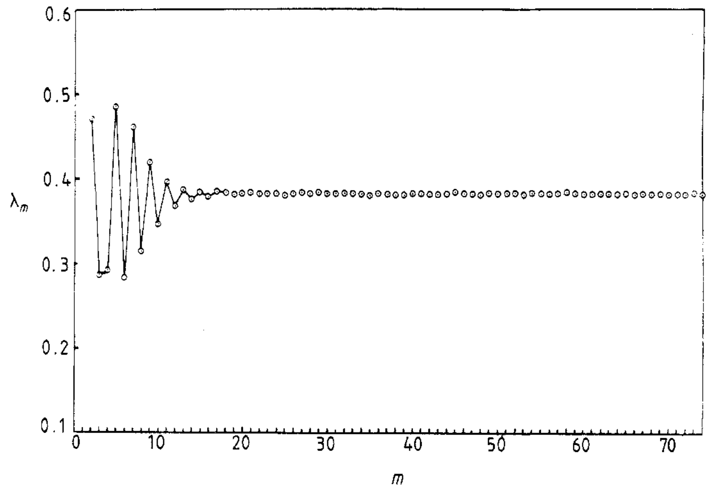

Figure 4. Plot of Lyapounov numbers λ_m corresponding to the sequence $\{\phi_m\}$ (5). The confinement of λ_m around an average value is obvious. Note the initial oscillations in λ_m. The lines joining the successive points serve as a visual aid.

oscillations are large but quickly die out and the successive Lyapounov numbers cluster around λ_∞. As the sequence ϕ_m is scanned to approach the target map $\phi(x)$, after a finite m value ($m > 1$) the maps ϕ_m and the target map $\phi(x)$ share the property of generating chaotic dynamics. Again, after a finite value of m ($m > 18$) the chaotic character of the system dynamics is seen to be almost independent of m, apart from negligible fluctuations.

We now compare the chaotic dynamics due to the supercritical Newton iteration and the tent map with that due to $\{\phi_m(x)\}$ of (5). As seen before, the supercritical Newton iteration can be replaced by the Bernoulli shift, whose Lyapounov number is a constant independent of any parameter. Thus a fully chaotic dynamics is generated and the 'amount of chaos' does not depend on any parameter. For the tent map, the Lyapounov number increases logarithmically with the parameter a, becomes positive for $a > \frac{1}{2}$ and tends to the value for the Bernoulli shift as $a \to 1$. We can now see that the parameter dependence obtained for the maps $\{\phi_m(x)\}$, as given in the above paragraph and figure 4, is essentially different from that obtained for the above two maps. Thus it does not seem possible to reduce the dynamics due to $\{\phi_m(x)\}$ to that due to any one of the above maps using a transformation.

In order to check on the possible dependence of the Lyapounov number on the initial conditions, we computed 100 Lyapounov numbers for the target map $\phi(x)$ (6) using 100 different values for x_0 in (2), generated by a uniform random number generator over the open interval $(0, 1)$. These turn out to be constant with average value $\lambda_\infty = 0.382\,91$ and standard deviation $0.858\,26 \times 10^{-3}$, both rounded to five digits.

The most important finding of these computer experiments is the observed dependence of Lyapounov numbers on the parameter m labelling the maps $\{f_m(x)\}$ and $\{\phi_m(x)\}$ in (3) and (5). This dependence can be summarised as follows.

(a) For m exceeding a finite value, $\lambda_m > 0$.

(b) For m exceeding a finite value, λ_m becomes almost independent of m and clusters in a small neighbourhood around λ_∞ which is the Lyapounov number for the target map.

These observations indicate that the sensitive dependence on initial conditions is a dynamical invariant for $\{f_m(x)\}$ and $\{\phi_m(x)\}$ as the corresponding target map is approached through these maps. However, since our computer experiments do not explicitly use any norm under which $\{f_m(x)\}$ or $\{\phi_m(x)\}$ converge, the experiments are not indicative of the exact role played by the type of convergence in generating the observed dynamics with the above properties. Thus an immediate problem is to relate the characteristics of maps and the types of convergence to the observed dynamics. This is analogous to the problem of obtaining general features of corresponding dynamics from the characteristics of the unimodal maps [5, 6]. In this context we wish to point to the fact that the maps $\{f_m(x)\}$ are bimodal while the maps $\{\phi_m(x)\}$ are not even unimodal, although both share the qualitative features regarding the dependence of Lyapounov number on their respective parameters. We feel that the maps considered here point to a new scenario for chaos. At any rate, we think that the work presented in this paper enables one to meaningfully ask the following question. Does there exist a class of maps for which the sensitive dependence on initial conditions is a dynamical invariant in the above sense? And if it does exist, what are its essential characteristics?

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